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Kac-Moody Symmetries of IIB Supergravity

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Abstract
We formulate the bosonic sector of IIB supergravity as a non-linear realisation. We show that this non-linear realisation contains the Borel subalgebras of SL(11) and E_7 and argue that it can be enlarged so as to be based on the rank eleven Kac-Moody algebra E_{11}

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0. Introduction

It is well known since the work of Nahm [1] that there is only one supergravity theory in eleven dimensions, but there are two maximally supersymmetric theories in ten dimensions. By compactifying one of the dimensions of the eleven dimensional supergravity theory [2], one finds one of the maximal supergravity theories in ten dimensions. This theory has a supersymmetry whose 32 component supercharge is a Majorana spinor and it has become known as the IIA supergravity theory [3]. The other maximal supergravity theory in ten dimensions, called IIB supergravity [4,5,6], has two supercharges that are two 16 component Majorana-Weyl spinors of same chirality. By virtue of their high degree of supersymmetry and corresponding uniqueness, the maximal supergravity theories in ten dimensions are the low energy effective actions for the corresponding superstring theories. As such, they can provide important information about these string theories including some aspects of their non-perturbative behaviour.

One of the most remarkable features of supergravity theories is that the scalars they contain always occur in a coset structure. While this can be viewed as a consequence of supersymmetry, the groups that occur in these cosets are rather mysterious. The two most studied examples are perhaps the $E_7/SU(8)$ [7] of the maximal supergravity in four dimensions and the $SU(1,1)/U(1)$ [4] of the ten dimensional IIB theory. It has been conjectured [8] that the symmetries found in these cosets are symmetries of the associated non-perturbative string theory.

The coset construction was extended [9] to include the gauge fields of supergravity theories. This method used generators that were inert under Lorentz transformations and, as such, it is difficult to extend this method to include either gravity or the fermions. However, this construction did include the gauge and scalar fields as well as their duals, and as a consequence the equations of motion for these fields could be expressed as a generalised self-duality condition. This formulation was given for eleven dimensional supergravity, all its reductions to four dimensions as well as for the IIB theory [9].

Recently [10], it was shown that the entire bosonic sectors of eleven dimensional and ten dimensional IIA supergravity theories could be formulated as non-linear realisations. In this way of proceeding gravity and the gauge fields appeared on an equal footing and one could hope to see the full symmetries of supergravity theories.

Here we shall confirm the conjecture in references [10,11] that the entire bosonic sector of ten dimensional IIB supergravity can also be formulated as a non-linear realisation. The formulation of the IIB theory we find is one in which all the degree of freedom of the theory, except for the graviton and the four form gauge field which satisfies a self-duality condition, are described by a gauge field and its dual gauge field. The equations of motion just relate the two field strengths using Hodge duality. This construction is carried out in section one.

In reference [11] it was argued that the non-linear realisation of eleven dimensional supergravity given in [10] could be extended so that eleven dimensional supergravity could be formulated as a non-linear realisation based on a rank eleven Kac-Moody algebra called E_{11} in [11]. In section two we assume that a similar enlargement is possible for the non-linear realisation of the IIB theory given in section one. By first showing that the non-linear realisation of section one contains the Borel subalgebras of SL(11) and E_7, we identify this Kac-Moody algebra to be also E_{11}. We then outline how this non-linear realisation can be enlarged to contain the Borel subalgebra of E_8. This last step implies that the gravitational degrees of freedom must be described by by two fields which are related by duality. Finally, we discuss the consequences of these results for the relationship between the IIA and IIB theories and M theory.

1. IIB Supergravity as a Non-linear Realisation

The degrees of freedom of this theory consists of the graviton, denoted by the field ${h_a}^b$, two scalars A^1 and A^2 corresponding to the dilaton and the axion respectively, two 2-form gauge potentials $A^1_{a_1a_2}$ and $A^2_{a_1a_2}$ and a 4-form gauge potential $A_{a_1\ldots a_4}$. The graviton belongs to the NS-NS sector in the associated IIB string theory as do the other fields if they carry the superscript one. The fields carrying the superscript two belong to the R-R sector.

As in references [9] and [10], we introduce dual fields for all the fields except for the graviton and the four form gauge field, whose field strength satisfies a type of self-dual condition. The complete field content is then given by the set

$$ {h_a}^b, A^s,\ A^s_{c_1c_2},\ A^2_{c_1\ldots c_4},,\ A^s_{c_1\ldots c_6},\ A^s_{c_1\ldots c_8}, \tag{1.1}$$

where s can take value 1 or 2 corresponding to the sectors. Each of the above fields is to be a Goldstone boson and as such we introduce a corresponding set of generators which is then given by

$$ {K^a}_b,\ R_s,\ R_s^{c_1c_2},\ R_2^{c_1\cdots c_4},\ R_s^{c_1\cdots c_6},\ R_s^{c_1\cdots c_8}, \qquad s=1,2 \tag{1.2}$$

We also include the momentum generator P_c which introduces space-time into the group element.

We take the generators to obey the following relations:

$$\begin{gathered}
[{K^a}_b, {K^c}_d] = \delta^c_b {K^a}_d - \delta^a_d {K^c}_b, \quad [{K^a}_b, P_c] = -\delta^a_c P_b, \\
[{K^a}_b, R_s^{c_1\cdots c_p}] = \delta^{c_1}_b R_s^{ac_2\cdots c_p} + \cdots, \\
[R_{s_1}^{c_1\cdots c_p}, R_{s_2}^{c_1\cdots c_q}] = c^{s_1,s_2}_{p,q} R^{c_1\cdots c_{p+q}}_{s(s_1,s_2)}, \\
[R_1, R_s^{c_1\cdots c_p}] = d^s_p R_s^{c_1\cdots c_p}, \quad [R_2, R_{s_1}^{c_1\cdots c_p}] = \tilde{d}^s_p R^{c_1\cdots c_p}_{s(2,s_1)},
\end{gathered} \tag{1.3}$$

where $+\cdots$ means the appropriate anti-symmetrisations. The generators ${K^a}_b$ satisfy the commutation relations of $GL(10,\mathbf{R})$. In the third line the superscript s depends on the fields in the commutator and we have therefore written $s=s(s_1,s_2)$. This function satisfies the properties $s(1,1)=s(2,2)=1, \quad s(1,2)=s(2,1)=2$. In the last line we have split the scalar commutators into those for the dilaton (superscript 1) with coefficient d^s_p, and the axion (subscript 2) with coefficient $\tilde{d}^s_p$. One can see that in the commutator the dilaton is sector preserving while the axion changes the sector of the other generator. The Jacobi identity implies the following relations among the constants.

$$ c^{s_2,s_3}_{q,r} c^{s_1,s(s_2,s_3)}_{p,q+r} = c^{s_1,s_2}_{p,q} c^{s(s_1,s_2),s_3}_{p+q,r} + c^{s_1,s_3}_{p,r} c^{s_2,s(s_1,s_3)}_{q,p+r}, \tag{1.4}$$

where

$$ c^{2,s_2}_{0,q} \equiv \tilde{d}^{s_2}_p $$

and

$$ (d^{s_3}_q + d^{s_1}_p - d^{s(s_1,s_3)}_{p+q}) c^{s_1,s_3}_{p,q} = 0. \tag{1.5}$$

The constants in the above commutation relations are taken to be:

$$d_2^1 = d_6^2 = -d_2^2 = -d_6^1 = \frac{1}{2},\ d_0^2 = -d_8^2 = -1, \tag{1.6}$$

$$c_{2,2}^{1,2} = -c_{2,2}^{2,1} = -1, \quad c_{2,4}^{2,2} = -c_{2,4}^{1,2} = 4, \quad c_{2,6}^{1,2} = 1, \quad c_{2,6}^{1,1} = -c_{2,6}^{2,2} = \frac{1}{2}$$

$$\tilde{d}_2^1 = -\tilde{d}_6^2 = -\tilde{d}_8^2 = 1, \quad \tilde{d}_2^2 = \tilde{d}_6^1 = \tilde{d}_8^1 = 0 \tag{1.7}$$

All not mentioned coefficients are zero. One can verify that they do indeed satisfy the Jacobi relations. We denote the above algebra by G_{IIB}.

The algebra G_{IIB} possesses the Lorentz algebra as a subalgebra. The generators J_{ab} of the latter are given by the anti-symmetric part of the $K^a{}_b$ generators, i e: $J_{ab} = K_{ab} - K_{ba}$, where the indices are lowered and raised with the Minkowski metric. We will show that IIB supergravity can be described as a nonlinear representation of the group G_{IIB} taking the Lorentz group as the local subgroup. The general element of G_{IIB} can be written as

$$g = \exp(x^\mu P_\mu)\exp(h_a{}^b K^a{}_b) g_A \equiv g_h g_A, \tag{1.8}$$

where

$$g_A = e^{(1/8!)A^2_{a_1\cdots a_8} R_2^{a_1\cdots a_8}} e^{(1/8!)A^1_{a_1\cdots a_8} R_1^{a_1\cdots a_8}} e^{(1/6!)(A^2_{a_1\cdots a_6} R_2^{a_1\cdots a_6} + A^1_{a_1\cdots a_6} R_1^{a_1\cdots a_6})}$$

$$\times e^{(1/4!)A^2_{a_1\cdots a_4} R_2^{a_1\cdots a_4}} e^{(1/2!)(A^2_{a_1 a_2} R_2^{a_1 a_2} + A^1_{a_1 a_2} R_1^{a_1 a_2})} e^{A^2 R_2} e^{A^1 R_1}. \tag{1.9}$$

For easier identification with the known literature in what follows below we will often relabel $A^1 = \sigma$ and $A^2 = \chi$.

Following the standard procedure of non-linear realisations we demand that the theory be invariant under

$$g \to g_0 g h^{-1}, \tag{1.10}$$

where g_0 is an element from the whole group G_{IIB} and is a rigid transformation while h is a local Lorentz transformation.

We now calculate the Maurer-Cartan form

$$\mathcal{V} = g^{-1}dg - \omega \tag{1.11}$$

in the presence of the Lorentz connection $\omega = \frac{1}{2}dx^\mu \omega_{\mu b}{}^a J^b{}_a$, which transforms as

$$\omega \to h\omega h^{-1} + hdh^{-1}. \tag{1.12}$$

As a result, $\mathcal{V}$ transforms as

$$\mathcal{V} \to h\mathcal{V}h^{-1}. \tag{1.13}$$

Writing $\mathcal{V}$ in the form

$$\mathcal{V} = (g_h^{-1}dg_h) + (g_A^{-1}dg_A + g_A^{-1}(g_h^{-1}dg_h)g_A - g_h^{-1}dg_h), \tag{1.14}$$

using the relations

$$e^{-A}de^{A} = dA - \frac{1}{2}[A, dA] + \frac{1}{6}[A, [A, dA]] - \frac{1}{24}[A, [A, [A, dA]]] + \cdots,$$

$$e^{-A}Be^{A} = B - [A, B] + \frac{1}{2}[A, [A, B]] + \cdots, \tag{1.15}$$

and the commutation relations of the G_{IIB} algebra and we find that

$$\mathcal{V} \equiv dx^{\mu}(e_{\mu}{}^{a}P_{a} + dx^{\mu}\Omega_{\mu a}{}^{b}K^{a}{}_{b}) + dx^{\mu}(\sum_{p=1}^{8}\frac{1}{p!}e^{-d^{s}_{p-1}\sigma}\tilde{D}_{\mu}A^{s}_{a_1\cdots a_p}R_{s}^{a_1\cdots a_p}), \tag{1.16}$$

where

$$e_{\mu}{}^{a} = (e^{h})_{\mu}{}^{a}, \tag{1.17}$$

$$\Omega_{ab}{}^{c} \equiv (e^{-1})_{a}{}^{\mu}(e^{-1}\partial_{\mu}e)_{b}{}^{c} - \omega_{ab}{}^{c}, \tag{1.18}$$

and the definition of $\tilde{D}_{\mu}A_{a_1\cdots a_p}$ will be given below.

The IIB supergravity theory is the non-linear realisation of the group that is the closure of the G_{IIB} algebra given above with the ten dimensional conformal algebra. However, rather than working with this infinite dimensional group we first construct the Cartan forms of the G_{IIB} algebra, as above, and then take only such combinations of these that can be rewritten in terms of Cartan forms of the conformal group. This procedure was described in detail in reference [10] and we will simply state the results of this method when applied to the G_{IIB} algebra. In the gravity sector we adopt the unique constraint

$$\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0, \tag{1.19}$$

which gives the usual expression for the spin connection in terms of the vielbein. The only objects which are Lorentz covariant and therefore covariant under the full non-linear realisation composed out of the closure of the conformal and G_{IIB} algebras are the Riemann tensor composed out of the spin-connection in the usual way and the completely anti-symmetrised derivatives $e^{-d^{s}_{p-1}\sigma}\tilde{D}_{[a_1}A_{a_2\cdots a_p]}$. The latter are denoted by

$$\tilde{F}^{s}_{a_1\cdots a_p} = pe^{-d^{s}_{p-1}\sigma}\tilde{D}_{[a_1}A_{a_2\cdots a_p]}. \tag{1.20}$$

We observe that these expressions begin with the field strength of the gauge fields as they should. The explicit expressions for these objects, whose calculation was explained above, are then given for the scalars by

$$\tilde{F}^{1}_{a} = \tilde{D}_{a}\sigma, \qquad \tilde{F}^{2}_{a} = e^{\sigma}\tilde{D}_{a}\chi, \tag{1.21}$$

for the 2-index fields:

$$\tilde{F}^{1}_{a_1a_2a_3} = 3e^{-\frac{1}{2}\sigma}\tilde{D}_{[a_1}A^{1}_{a_2a_3]}, \quad \tilde{F}^{2}_{a_1a_2a_3} = 3e^{\frac{1}{2}\sigma}(\tilde{D}_{[a_1}A^{2}_{a_2a_3]} - \chi\tilde{D}_{[a_1}A^{1}_{a_2a_3]}), \tag{1.22}$$

for the 4-index field

$$\tilde{F}_{a_1\cdots a_5} = 5(\tilde{D}_{[a_1}A_{a_2\cdots a_5]} + 3A^{1}_{[a_1a_2}\tilde{D}_{a_3}A^{2}_{a_4a_5]} - 3A^{2}_{[a_1a_2}\tilde{D}_{a_3}A^{1}_{a_4a_5]}), \tag{1.23}$$

for the 6-index fields

$$\tilde F^2_{a_1\cdots a_7} = 7e^{-\frac{1}{2}\sigma}\left(\tilde D_{[a_1}A^2_{a_2\cdots a_7]} + 60A^1_{[a_1a_2}(\tilde D_{a_3}A^2_{a_4\cdots a_7]} + A^1_{a_3a_4}\tilde D_{a_5}A^2_{a_6a_7]} - A^2_{a_3a_4}\tilde D_{a_5}A^1_{a_6a_7]})\right), \tag{1.24}$$

$$\tilde F^1_{a_1\cdots a_7} = 7e^{\frac{1}{2}\sigma}\left(\tilde D_{[a_1}A^1_{a_2\cdots a_7]} - 60A^2_{[a_1a_2}(\tilde D_{a_3}A^2_{a_4\cdots a_7]} + A^1_{a_3a_4}\tilde D_{a_5}A^2_{a_6a_7]} - A^2_{a_3a_4}\tilde D_{a_5}A^1_{a_6a_7]})\right)$$
$$+7e^{\frac{1}{2}\sigma}\chi\left(\tilde D_{[a_1}A^2_{a_2\cdots a_7]} + 60A^1_{[a_1a_2}(\tilde D_{a_3}A^2_{a_4\cdots a_7]} + A^1_{a_3a_4}\tilde D_{a_5}A^2_{a_6a_7]} - A^2_{a_3a_4}\tilde D_{a_5}A^1_{a_6a_7]})\right), \tag{1.25}$$

and finally for the 8-index fields

$$\tilde F^1_{a_1\cdots a_9} = 9\,(\,\tilde D_{[a_1}A^1_{a_2\cdots a_9]} - 7\cdot 2A^1_{[a_1a_2}\,(\,\tilde D_{a_3}A^1_{a_4\cdots a_9]} - 6\cdot 5A^2_{a_3a_4}(\tilde D_{a_5}A^2_{a_6\cdots a_9]}$$
$$+\frac{1}{2}A^1_{a_5a_6}\tilde D_{a_7}A^2_{a_8a_9]} - \frac{1}{2}A^2_{a_5a_6}\tilde D_{a_7}A^1_{a_8a_9]})\,)\,) + 7\cdot 2A^2_{[a_1a_2}\,(\,\tilde D_{a_3}A^2_{a_4\cdots a_9]} + 6\cdot 5A^1_{a_3a_4}(\tilde D_{a_5}A^2_{a_6\cdots a_9]}$$
$$+\frac{1}{2}A^1_{a_5a_6}\tilde D_{a_7}A^2_{a_8a_9]} - \frac{1}{2}A^2_{a_5a_6}\tilde D_{a_7}A^1_{a_8a_9]})\,)\,)\,) + 9\chi\,(\,\tilde D_{[a_1}A^2_{a_2\cdots a_9]}$$
$$-7\cdot 4A^1_{[a_1a_2}\,(\,\tilde D_{a_3}A^2_{a_4\cdots a_9]} + 6\cdot 5A^1_{a_3a_4}(\tilde D_{a_5}A^2_{a_6\cdots a_9]} + \frac{1}{2}A^1_{a_5a_6}\tilde D_{a_7}A^2_{a_8a_9]} - \frac{1}{2}A^2_{a_5a_6}\tilde D_{a_7}A^1_{a_8a_9]})\,)\,) \tag{1.26}$$

and

$$\tilde F^2_{a_1\cdots a_9} = 9e^{-\sigma}\,(\,\tilde D_{[a_1}A^2_{a_2\cdots a_9]} - 7\cdot 4A^1_{[a_1a_2}\,(\,\tilde D_{a_3}A^2_{a_4\cdots a_9]} + 6\cdot 5A^1_{a_3a_4}(\tilde D_{a_5}A^2_{a_6\cdots a_9]}$$
$$+\frac{1}{2}A^1_{a_5a_6}\tilde D_{a_7}A^2_{a_8a_9]} - \frac{1}{2}A^2_{a_5a_6}\tilde D_{a_7}A^1_{a_8a_9]})\,)\,)\,. \tag{1.27}$$

The equations of motion can only be constructed from the spin connection and the covariant objects $\tilde F^s_{a_1\ldots a_p}$ and can only be

$$\tilde F^{1\mu\nu\rho} = \frac{1}{7!}\epsilon^{\mu\nu\rho\mu_1\cdots\mu_7}\tilde F^1_{\mu_1\cdots\mu_7}, \quad \tilde F^{2\mu\nu\rho} = \frac{1}{7!}\epsilon^{\mu\nu\rho\mu_1\cdots\mu_7}\tilde F^2_{\mu_1\cdots\mu_7}, \tag{1.28}$$

$$\tilde F^{1\mu} = \frac{1}{9!}\epsilon^{\mu\mu_1\cdots\mu_9}\tilde F^1_{\mu_1\cdots\mu_9}, \quad \tilde F^{2\mu} = \frac{1}{9!}\epsilon^{\mu\mu_1\cdots\mu_9}\tilde F^2_{\mu_1\cdots\mu_9}, \tag{1.29}$$

The remaining equation of motion is that for the vielbein and is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - [\frac{1}{2}\partial_{(\mu}\sigma\partial_{\nu)}\sigma + \frac{1}{2}e^{2\sigma}\partial_{(\mu}\chi\partial_{\nu)}\chi - \frac{1}{4}g_{\mu\nu}\left(\partial_\mu\sigma\partial^\mu\sigma + e^{2\sigma}\partial_\mu\chi\partial^\mu\chi\right) +$$
$$-\frac{1}{6}\tilde F_{\mu_1\cdots\mu_4\mu}\tilde F^{\mu_1\cdots\mu_4}{}_\nu - \frac{1}{16}e^\sigma\tilde F^{2\mu_1\mu_2}_{(\mu}\tilde F^2_{\nu)\mu_1\mu_2} - \frac{1}{16}e^{-\sigma}\tilde F^{1\mu_1\mu_2}_{(\mu}\tilde F^1_{\nu)\mu_1\mu_2} +$$
$$+\frac{1}{96}g_{\mu\nu}(e^\sigma\tilde F^{2\mu_1\mu_2\mu_3}\tilde F^2_{\mu_1\mu_2\mu_3} + e^{-\sigma}\tilde F^{1\mu_1\mu_2\mu_3}\tilde F^1_{\mu_1\mu_2\mu_3})] = 0 \tag{1.30}$$

The value of the constants in front of the field strength squared terms can only be fixed by considering the full non-linear realisation of the IIB theory that includes the fermionic sector of the theory or the Kac-Moody groups considered later in this paper. In the above equation we have fixed the values of these constants to their correct values.

We can obtain the more standard second order equations of IIB supergravity in terms of the original fields without their duals by differentiating the equations (1.28) and (1.29) and using the Bianchi identities of the dual field strengths. For example, we can rewrite the second equation in equation (1.28) as $\tilde{F}^2_{\mu\nu\rho} = \frac{1}{7!}\epsilon_{\mu\nu\rho\mu_1\cdots\mu_7}\tilde{F}^{2\mu_1\cdots\mu_7}$, inserting expressions (1.22) and (1.24) for these field strengths into this equation, bringing the exponential of σ to the left hand side, we find that differentiating the whole expression yields the equation

$$\partial^\rho(e^\sigma \tilde{F}^2_{\mu\nu\rho}) = \frac{2}{3}\tilde{F}^{1\rho\mu_1\mu_2}G_{\mu\nu\rho\mu_1\mu_2}. \tag{1.31}$$

Similarly we find that the other equations imply that

$$\partial^\rho(e^{-\sigma}\tilde{F}^1_{\mu\nu\rho}) = -\frac{2}{3}G_{\mu\nu\rho\lambda\sigma}\tilde{F}^{2\rho\lambda\sigma} + \partial^\rho\chi(e^\sigma\tilde{F}^2_{\mu\nu\rho}),$$

$$\partial^\mu(e^{2\sigma}\partial_\mu\chi) = -\frac{1}{6}e^\sigma\tilde{F}^{1\mu\nu\rho}\tilde{F}^2_{\mu\nu\rho},$$

$$\partial^\mu(\partial_\mu\sigma) = e^{2\sigma}\partial^\mu\chi\partial_\mu\chi + \frac{1}{12}e^\sigma\tilde{F}^{2\mu\nu\rho}\tilde{F}^2_{\mu\nu\rho} - \frac{1}{12}e^{-\sigma}\tilde{F}^{1\mu\nu\rho}\tilde{F}^1_{\mu\nu\rho}. \tag{1.32}$$

Since the field strength of the four form gauge field is self-dual we leave this equation to be first order in derivatives.

The IIB supergravity theory was first discovered [4,5,6] in a formulation in which the scalars belong to the coset $SU(1,1)/U(1)$. In this formulation the field content consists of the graviton, a complex two form gauge fields $A^s_{a_1a_2}$, a real four form $A_{a_1\ldots a_4}$ and two scalars denoted by the complex field ϕ which belong to the coset $SU(1,1)/U(1)$. The rewriting of the IIB supergravity theory in terms of an SL(2,$\mathbf{R}$)/SO(2) coset was given in reference [12] and was reviewed in reference [13]. The equations of motion given above are precisely those found in this latter formuation.

2. E_{11} and IIB Supergravity

In reference [11] it was argued that eleven dimensional supergravity was invariant under a Kac-Moody algebra that was identified to be E_{11}. We refer the reader to this paper for a discussion of how the non-linear realisation of eleven dimensional supergravity given in reference [10] might be extended to incorporate such a large algebra by using an alternative formulation of eleven dimensional supergravity and increasing the size of the local subgroup. The same group was identified as a symmetry of the IIA supergravity theory. In this section, we will assume that the non-linear realisation of the IIB supergravity theory given above can be similarly enlarged to a non-linear realisation of a Kac-Moody algebra. We will show that this algebra is also E_{11}.

The proposed Kac-Moody algebra of the IIB theory must contain the algebra denoted G_{IIB} above and given in equations (1.3) to (1.7). In the non-linear realisation discussed above the local subgroup is taken to be the ten dimensional Lorentz group and so all the remaining generators in G_{IIB} are coset generators and as such correspond to fields in IIB supergravity. In the enlarged non-linear realisation based on a Kac-Moody algebra, the local subgroup is taken to be that invariant under the Cartan involution and as a result the coset representatives can be written as exponentials of the Cartan subalgebra and positive root generators of the Kac-Moody algebra. Consequently, all the generators of G_{IIB}, except the negative root generators of SL(10), must be included in the Cartan subalgebra and positive root generators of the Kac-Moody algebra. A set of commuting generators of G_{IIB} can be taken to be

$$K^a{}_a,\ a = 1, \dots 10, \quad \text{and} \quad R_1 \tag{2.1}$$

and these may be taken to belong to the Cartan subalgebra of the Kac-Moody algebra. We also observe that the remaining generators of G_{IIB}, except the negative root generators of SL(10), can be generated by taking multiple commutators of the generators

$$K^a{}_{a+1},\ a = 1, \dots 9, \quad R_2 \quad \text{and} \quad R_1^{910}. \tag{2.2}$$

We may identify these as positive simple root generators of the Kac-Moody algebra. Thus we are seeking a rank eleven Kac-Moody algebra.

Calculating the commutator of the positive simple root and Cartan sub-algebra generators leads to the Cartan matrix from which we can uniquely identify the Kac-Moody algebra. However, working with only the Cartan sub-algebra and simple positive root generators -that is without the negative simple root generators- does not automatically encode the particular basis for the Cartan sub-algebra that satisfies the Chevalley relations and hence produces the correct Cartan matrix. However, we must use a basis that leads to an acceptable Cartan matrix, that is one which satisfies the correct properties to be associated with a Kac-Moody algebra. Even taking this into account, the choice of the basis is not free from ambiguity. As in reference [11] this ambiguity may be resolved by identifying appropriate subgroups and so we first carry out this step.

The non-linear realisation of the IIB supergravity theory given above is obviously invariant under SL(10) generated by $K^a{}_b,\ a, b = 1, \dots 10$, but as we now show it also is invariant under the Borel subgroup of SL(11). One can verify, using equations (1.3) to (1.7) that the generators

$$\hat{K}^a{}_b, \quad \hat{K}^a{}_{10} = R_1^{a10}, \quad \hat{K}^a{}_{11} = -R_2^{a10},$$

$$\hat{K}^{10}{}_{11} = R_2, \quad \hat{K}^{11}{}_{11} = R_1 + \frac{1}{4}\sum_{a=1}^{10} K^a{}_a - K^{10}{}_{10}, \quad a, b = 1, \dots 9, \tag{2.3}$$

do indeed obey the commutation relations for the Borel subgroup of SL(11). We note that this SL(11) only coincides with the SL(9) subgroup of the obvious SL(10) group, that is for the generators which carry the indices $a, b = 1, \dots, 9$.

The G_{IIB} algebra also contains the rank three anti-symmetric representation of the Borel subgroup of SL(11). More precisely, if we identify

$$X^{a_1a_2a_3} = R_2^{10a_1a_2a_3}, X^{10a_1a_2} = R_2^{a_1a_2}, X^{11a_1a_2} = R_1^{a_1a_2}, X^{a_1 1011} = K^{a_1}{}_{10}, a_1, ., a_3 = 1, \dots, 9 \tag{2.4}$$

then equations (1.3) to (1.7) imply the relation

$$[\hat{K}^{\hat{a}}{}_{\hat{b}}, X^{\hat{c}_1\hat{c}_2\hat{c}_3}] = 3\delta_{\hat{b}}^{[\hat{c}_1} X^{|\hat{a}|\hat{c}_2\hat{c}_3]},\ a \le b,\ \hat{a}, \hat{b}, \hat{c}_1, \hat{c}_2, \hat{c}_3 = 1, \dots, 11 \tag{2.5}$$

To identify the Borel subgroup of E_7 contained in G_{IIB} we consider only the generators whose indices take the values $i, j, \dots = 5, \dots 10$. We search for the formulation of the E_7 algebra which has its SL(7) subgroup manifest. The generators $\hat{K}^{\hat{i}}{}_{\hat{j}},\ \hat{i} \le \hat{j}$ generate the 27 dimensional Borel sub-algebra of this SL(7) together with a U(1) factor. The generators $X^{\hat{i}_1\hat{i}_2\hat{i}_3}$ belong to the 35 dimensional third rank anti-symmetric representation of SL(7). Calculating the commutator of the later generators we find that

$$[X^{\hat{i}_1\hat{i}_2\hat{i}_3}, X^{\hat{i}_4\hat{i}_5\hat{i}_6}] = 2\epsilon^{\hat{i}_1\ldots\hat{i}_6\hat{k}} S_{\hat{k}} \tag{2.6}$$

where the 7 generators S_k are given by

$$S_i = \frac{1}{2.4!}\epsilon_{1011ij_1\ldots j_4} R_2^{j_1\ldots j_4},\ S_{10} = \frac{2}{5!}\epsilon_{1011j_1\ldots j_5} R_2^{j_1\ldots j_5 10},$$

$$S_{11} = \frac{2}{5!}\epsilon_{1011j_1\ldots j_5} R_1^{j_1\ldots j_5 10},\ j_1\ldots = 5,\ldots,9. \tag{2.7}$$

It is then straightforward to verify that the generators

$$\hat{K}^{\hat{i}}{}_{\hat{j}},\ \hat{i}\le\hat{j},\ X^{\hat{i}_1\hat{i}_2\hat{i}_3}, S_{\hat{k}} \tag{2.8}$$

satisfy all the remaining commutation relations of the 70 dimensional Borel subgroup of E_7. One can also check that for this restriction of the G_{IIB} that there are no other generators contained in G_{IIB} other than the negative root generators of SL(7). This demonstrates that IIB supergravity is invariant under the Borel subgroups of SL(11) and E_7. However, we expect that it is also invariant under the full SL(11) and E_7 groups, the missing generators forming part of an enlarged local subgroup.

Given the above identification of the SL(11) and E_7 groups in the proposed rank eleven Kac-Moody algebra we can finally identify this Kac-Moody Lie algebra. The simple positive root generators are given by

$$E_a = K^a{}_{a+1}, a = 1,\ldots 8,\ E_9 = R_1^{910},\ E_{10} = R_2,\ E_{11} = K^9{}_{10}. \tag{2.9}$$

These agree with the simple positive root generators of the SL(11) and E_7 subgroups found above. The basis of the Cartan subalgebra that leads to an acceptable Cartan matrix and agrees with the above identifications of the Cartan subalgebra elements of SL(11) and E_7 is given by

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1,\ldots,8,\ H_9 = K^9{}_9 + K^{10}{}_{10} + R_1 - \frac{1}{4}\sum_{a=1}^{11} K^a{}_a,$$

$$H_{10} = -2R_1,\ H_{11} = K^9{}_9 - K^{10}{}_{10} \tag{2.10}$$

One can verify that

$$[H_a, E_b] = A_{ab}E_b \tag{2.11}$$

where A_{ab} is the Cartan matrix for E_{11}. Hence we identify E_{11} as the Kac-Moody algebra that underlies IIB supergravity.

In reference [11] it was explained how one might enlarge the non-linear realisation of eleven dimensional supergravity such that it contained the E_{11} Kac-Moody algebra. In particular, it was shown how by adopting a first order formulation of gravity involving the usual metric and a dual field one could extend the non-linear realisation to include the Borel subgroup of E_8.

We refer the reader to this reference for the details of this procedure and we now outline the analogous steps for IIB supergravity.

We consider the restriction of the G_{IIB} algebra resulting from only considering generators with the indices $i, j = 4, \dots 10$. We will find the formulation of the E_8 algebra with the SL(8) symmetry manifest and so the generators will carry the indices $\hat{i}, \hat{j} = 4, \dots 11$. The Borel subgroup of this SL(8) is provided by the generators $\hat{K}^{\hat{i}}{}_{\hat{j}},\ \hat{i} \le \hat{j}$. Since this coincides with only the SL(6) subgroup of the obvious SL(8) we will have to treat the indices from $\hat{i}, \hat{j} = 10, 11$ differently from $\hat{i}, \hat{j} = 4, \dots 9$. The E_8 algebra possesses the commutation relations

$$[X^{\hat{i}_1\hat{i}_2\hat{i}_3}, X^{\hat{i}_4\hat{i}_5\hat{i}_6}] = \epsilon^{\hat{i}_1\hat{i}_2\hat{i}_3\hat{i}_4\hat{i}_5\hat{i}_6\hat{k}\hat{l}} S_{\hat{k}\hat{l}} \tag{2.12}$$

and

$$[S_{\hat{k}_1\hat{k}_2}, X^{\hat{i}_1\hat{i}_2\hat{i}_3}] = 3\delta^{[\hat{i}_1\hat{i}_2}_{\hat{k}_1\hat{k}_2} S^{\hat{i}_3]} \tag{2.13}$$

Calculating the commutators of the generators $X^{\hat{i}_1\hat{i}_2\hat{i}_3}$ of equation (2.4), we indeed find generators $S_{\hat{k}\hat{l}}$ with the exception that S_{1011} is absent. These new generators then obey equation (2.13) and lead to new generators $S^{\hat{k}}$ with the exception of $S^i,\ i = 4, \dots 9$.

The resolution of this dilemma is to modify the G_{IIB} algebra. We modify the commutation relation

$$[R_2^{a_1\dots a_6}, R_2^{a_7a_8}] = \frac{1}{2} R_1^{a_1\dots a_6a_7a_8} \tag{2.14}$$

to become

$$[R_2^{a_1\dots a_6}, R_2^{a_7a_8}] = \frac{1}{2} R_1^{a_1\dots a_6a_7a_8} + \tilde{R}_1^{a_1\dots a_6[a_7,a_8]}. \tag{2.15}$$

The new generator $\tilde{R}_1^{a_1\dots a_6a_7,a_8}$ is anti-symmetric in its first seven indices. The Jacobi identities then imply that this generator then also appears on the right-hand side of the relations

$$[R_1^{a_1\dots a_6}, R_1^{a_7a_8}] = -\frac{1}{2} R_1^{a_1\dots a_6a_7a_8} + \tilde{R}_1^{a_1\dots a_6[a_7,a_8]}. \tag{2.16}$$

and

$$[R_2^{a_1\dots a_4}, R_2^{a_5\dots a_8}] = 8\tilde{R}_1^{a_1a_2a_3a_4[a_5a_6a_7,a_8]}. \tag{2.17}$$

Re-evaluating the commutators of equations (2.12) and (2.13) with these modified relations we now find the missing generators of E_8 which are given by

$$S_{kl} = \frac{1}{4!}\epsilon_{kli_1\dots i_4} R_2^{i_1\dots i_4},\ S_{k10} = \frac{2}{5!}\epsilon_{ki_1\dots i_5} R_2^{i_1\dots i_5 10},$$

$$S_{k11} = -\frac{2}{5!}\epsilon_{ki_1\dots i_5} R_1^{i_1\dots i_5 10},\ S_{1011} = -\frac{1}{6!}\epsilon_{i_1\dots i_6}\tilde{R}_1^{i_1\dots i_6 10,10}, \tag{2.18}$$

and

$$S^k = -\frac{4}{5.5!}\epsilon_{i_1\dots i_6}\tilde{R}^{i_1\dots i_5 10[k,i_6]},\ S^{10} = -\frac{4}{6!}\epsilon_{i_1\dots i_6} R_1^{i_1\dots i_6},\ S^{11} = -\frac{4}{6!}\epsilon_{i_1\dots i_6} R_2^{i_1\dots i_6} \tag{2.19}$$

For the restriction of the restriction of the G_{IIB} algebra considered above that there are no other generators except those considered so far and these generate the Borel subalgebra of E_8. The 248 adjoint of E_8 decomposes into SL(8,R) representations as

$$248 = 1(\sum_{\hat{i}} K^{\hat{i}}{}_{\hat{i}}) + 63(K^{\hat{i}}{}_{\hat{j}}) + 56(X^{\hat{i}_1\hat{i}_2\hat{i}_3}) + \bar{28}(S_{\hat{k}\hat{l}}) + \bar{8}(S^{\hat{k}}) \tag{2.20}$$

as well as the negative roots $\bar{56} + 28 + 8$.

As explained in reference [11], the introduction of the extra generator $\tilde{R}_1^{a_1\dots a_6 a_7,a_8}$ in the G_{IIB} algebra implies the presence of an additional field $h_{a_1\dots a_7,a_8}$ which together with $h_a{}^b$ provides a first order formulation of gravity.

3. Discussion

In this paper we have shown that the bosonic sector of the IIB supergravity theory can be formulated as a non-linear realisation of an infinite dimensional algebra which is the closure of the conformal algebra and the algebra denoted above by G_{IIB}. This formulation includes the Borel subalgebra of SL(11) and E_7, but we argue that the non-linear realisation can be enlarged to include the Kac-Moody algebra E_{11}. We carry out the first step in this enlargement and show, by using a first order formulation of gravity involving two fields which are related by duality, that the algebra contains the Borel subalgebra of E_8.

It was perhaps not too surprising that the IIA supergravity theory has the same Kac-Moody algebra underlying it as the eleven dimensional supergravity theory as they are related by a reduction on a circle. However, IIB supergravity can not be obtained from eleven dimensional supergravity in a simple way and so the appearance of the same algebra is perhaps surprising. It is consistent with the idea expressed in [10,11] that M theory has an underlying E_{11} symmetry and that the maximal supergravity theories in eleven and ten dimensions appear as different manifestations of this symmetry. It is instructive to recall that non-linear realisations typically arise when a symmetry is spontaneously broken and it describes the theory controlling the low energy excitations. The local subgroup in the non-linear realisation corresponds to that part of the original symmetry that is preserved in the symmetry breaking. If a theory has different possible vacua one finds corresponding different low energy theories based on the same symmetry group, but with different local symmetry groups. This is precisely the picture we find with the maximal supergravities, they possess the same underlying group E_{11}, but have different local subgroups and so can be interpreted as different vacua of one theory, namely M theory. Indeed, from this perspective what distinguishes the IIA and IIB theory is the way the SL(11) subgroup is embedded in E_{11} and correspondingly what parts of it are in the local subgroups. The embedding is fixed by the occurance of the momentum generator which in turn gives rise to the space-time coordinates in the theory.

The different SL(11) embeddings have an SL(9) subgroups in common. This is consistent with the fact that IIA and IIB supergravity theories are the same when reduced to nine dimensions where the different embeddings give rise to the T duality transformations [12] between these two theories.

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